Mathematics in Action from Lisbon

Engagement with the Popularization and Communication of Mathematics

A.M. Eiró[1], S. Nápoles[2], J. F. Rodrigues[3] and J.N. Silva[4]

[1]Museu de Ciência and Faculdade de Ciências da Universidade de Lisboa
[2] FCUL (Faculdade de Ciências da Universidade de Lisboa), Departamento de Matemática
[3]FCUL/Centro de Matemática e Aplicações Fundamentais da Universidade de Lisboa
[4]FCUL/Centro Interuniversitário de História das Ciências e da Tecnologia and Associação *Ludus*

Introduction

Although not a new subject, the Popularization of Mathematics, viewed as an activity of "sharing mathematics with a wider public" and "encouraging people to be more active mathematically", is a relatively recent topic that motivated the fifth international study [1] of the International Commission on Mathematical Instruction and had seen important developments after the World Mathematical Year - WMY2000.

In this article we describe the engagement with the popularization and communication of mathematics that is being developed at the University of Lisbon, within the historical environment provided by its Museum of Science, which has a rich heritage. There are a significant number of initiatives open to society, from the educational community to the general public, and there has been a special collaboration with the Portuguese Mathematical Society since its foundation. Enhanced by the challenge of the WMY2000 and the project *Matemática em Acção*, developed by the *Centro de Matemática e Aplicações Fundamentais,* we describe several activities such as exhibitions, films, interactive applications and publications, that range from ruled surfaces, sundials and architecture to mathematical games, taking into account mathematics as an educational resource and exploring the multimedia and computational tools for communicating mathematics.

An Historic Environment

Mathematics has a strong relationship with its past and the role of mathematical objects in time and space is a very special one. The history of mathematics is a very useful tool for popularizing mathematics and for helping mathematicians and mathematics teachers build the correct image of their science which is a key for development and for facing the great challenges of the 21st century.


1

The Museum of Science of the University of Lisbon is located in a 19th century building in the historical part of Lisbon (Fig. 1). Its cultural heritage, however, dates from a Jesuits' school of the 17th century, with the Noviciate of Cotovia. The College of the Nobles was created in the Enlightenment, based on the reforming ideas of the Marquis of Pombal, and the Polytechnic School was created in 1837 following the liberal reforms of the 19th century. With the re-foundation of the University of Lisbon in 1911, the Polytechnic School was transformed into the Faculty of Sciences. In the 1980s, the Faculty initiated a move to new facilities providing an opportunity for the creation of the Museum of Science, in 1985 [2].

The first century of teaching mathematics, at the Polytechnic School and during the earlier decades of the Faculty of Sciences, was strongly associated with the teaching of astronomy and surveying sciences. This justified the creation, in 1875, of an astronomical observatory for training students [3], which was reconstructed in 1898 (Fig. 2). Among the professors that taught astronomy and mathematics we refer Filipe Folque (1800-1874) and José Sebastião e Silva (1914-1972), respectively.


2

Within this historic background and with a well-defined mission of collecting, preserving, and studying scientific instruments and scientific heritage in general, the Museum of Science aims to communicate science to broad audiences in astronomy, mathematics, physics and chemistry [4]. The Museum has established close collaborations with several institutions, in particular in mathematics, with the *Centro de Matemática e Aplicações Fundamentais* (CMAF) <w1> and the Department of Mathematics of the Faculty of Sciences, but also it has old and strong links with the Portuguese Mathematical Society (SPM). It is important to underline that the Museum collections held important historical archives related to mathematics, including relevant information about SPM. In fact, this Society was created in 1940 in the building where the Museum now stands.[1]

From Heritage to Society


3

With its rich historic legacy, the Museum is increasingly trying to integrate the collection (Fig. 3), history and archives into the communication programs developed for society. Over the past 25 years the Museum has held exhibitions, educational programs and special events, including scientific conferences, workshops, and many mathematics activities.

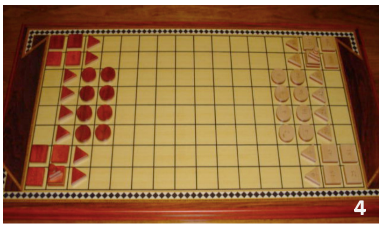
In 1987, for the bicentenary commemorations of José Anastácio da Cunha, a Portuguese mathematician and poet, an important temporary exhibition was held at the

National Library in Lisbon [5]. In December 1990, the commemoration of 50 years of the Portuguese Mathematical Society at the Museum of Science included an exhibition *Efervescência Matemática 1937-47*. This exhibition evolved into *Movimento Matemático 1937-1947,* a travelling version developed by SPM that started in 1997 at the *Museu da República*, with a catalogue published by the city of Lisbon [6].

In November 1997, an international conference on museums of science and technology, jointly organized by the Museum and the *Fundação Oriente*, was held in Arrábida, in a 16th century Franciscan monastery 30 km south of Lisbon. The proceedings [7] include two articles on the popularization of mathematics, a subject that was then creating an upsurge in museological interest.[2] Albrecht Beutelspacher revealed his pioneering project of a museum totally dedicated to mathematics [8] and Arala Chaves [9] showed the museological value of interactive exhibitions for mathematics, in particular, with the use of physical models and computer simulations.

Among the initiatives of preparation of the World Mathematical Year 2000 (WMY2000), the Museum of Science and CMAF helped organize a series of seven public lectures on mathematics and music, held at the *Fundação Calouste Gulbenkian*, in Lisbon, from January to July of 1998, that were published the next year [10]. Under the auspices of the European Mathematical Society, the Fourth Diderot Mathematical Forum on Mathematics and Music took place in December 1999 simultaneously in Lisbon, Paris and Vienna[3] [11]. In the aftermath of the WMY2000, the Museum of Science also hosted in December 2001 - February 2002 the itinerant exhibition *Beyond the Third Dimension*.

In 2007, the collaboration between the Museum and SPM was reinforced and two temporary exhibitions were produced: on the Portuguese mathematician Aniceto Monteiro [12] and on the history of *Portugaliae Mathematica*, the journal founded by Monteiro in 1937. Both exhibitions provided an opportunity to develop several activities of the mathematics club. A big mathematics fair was organized for the children´s day, attracting over one thousand kids. *Symmetries and mirrors*, a temporary exhibition produced by *Atractor* <w2> was displayed in the summer of 2007 and *Experiencing Mathematics* <w3>, an itinerant exhibition sponsored by UNESCO, ended its six month tour of Portugal at the Museum in the summer of 2008.



Two recent exhibitions were conceived and developed locally: *Mathematical Games throughout the Ages* and *Measuring the sky to rule the territory*. The first is an exhibition where unique replicas of historical board games are on display, coupled with hands-on boards for the public to use and play. It includes games like Pentalfa, Hex, Ludus Globi, Ludus Regularis, Rithmomachia (Fig. 4), Ouranomachia, Ludus Astronomorum and Stomachion. Still open, it was inaugurated in April 2008, when the Museum hosted the Board Game Studies Colloquium XI [13] organized by the *Associação Ludus*. During that colloquium, the first international report [14] was presented on the Portuguese Championship of Mathematical Games, co-organized by *Ludus*, SPM and APM (Mathematics Teachers Association) and sponsored by *Ciência Viva*, the national agency for science popularization. Since 2004, this championship has involved an increasing number of students aged 7 to 17, from all over the country, reaching 100,000 participants in the seventh championship in 2011.

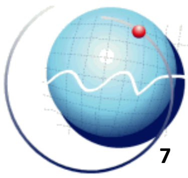
The second exhibition started during the International Year of Astronomy and following an international conference, organized at the Museum on the *History of Astronomy in Portugal* [15], in September 2009, in collaboration with CMAF and the *Centro Internacional de Matemática* (CIM). *Measuring the sky to rule the territory* (October 2009 - August 2010), was a historical exhibition on the teaching of Astronomy at the Polytechnic School, showing original astronomical instruments (Fig. 5) and documents, aimed at highlighting the connections between mathematics and astronomy [16].

Deriving from a previous exhibition about sundials, part of this exhibition explored the observation of stars for *Measuring time, measuring the world and measuring the sea* (Fig. 6), making a separate section. References to ancient Greek philosophers, as

Eratosthenes of Cyrene, Hipparchus of Nicaea and Aristarchus of Samos, as well as Portuguese 15th and 16th century sea voyages were interpreted in panels and organized in that separate section. This part became a travelling exhibition, which started in Óbidos, during the RPAM workshop in September 2010, is available for loan through SPM and is shown in secondary schools and other cultural institutions.

Activities for children and young people, schools and families have also been developed at the Museum of the University of Lisbon. In mathematics these activities explore logic, geometry and topology, by using puzzles, wood constructions, strategy games and board games[4]. Typical audiences are secondary school students, the majority between 10 and 15 years old. Participation in these activities, which cover all areas of science (physics, chemistry, mathematics, astronomy and also biology and geology), reached 48,000 in 2009. It is quite impressive that between mathematics and astronomy the total number of students reached 18,000, representing 37% of the total number of young visitors. And mathematics is one of the most popular subjects!

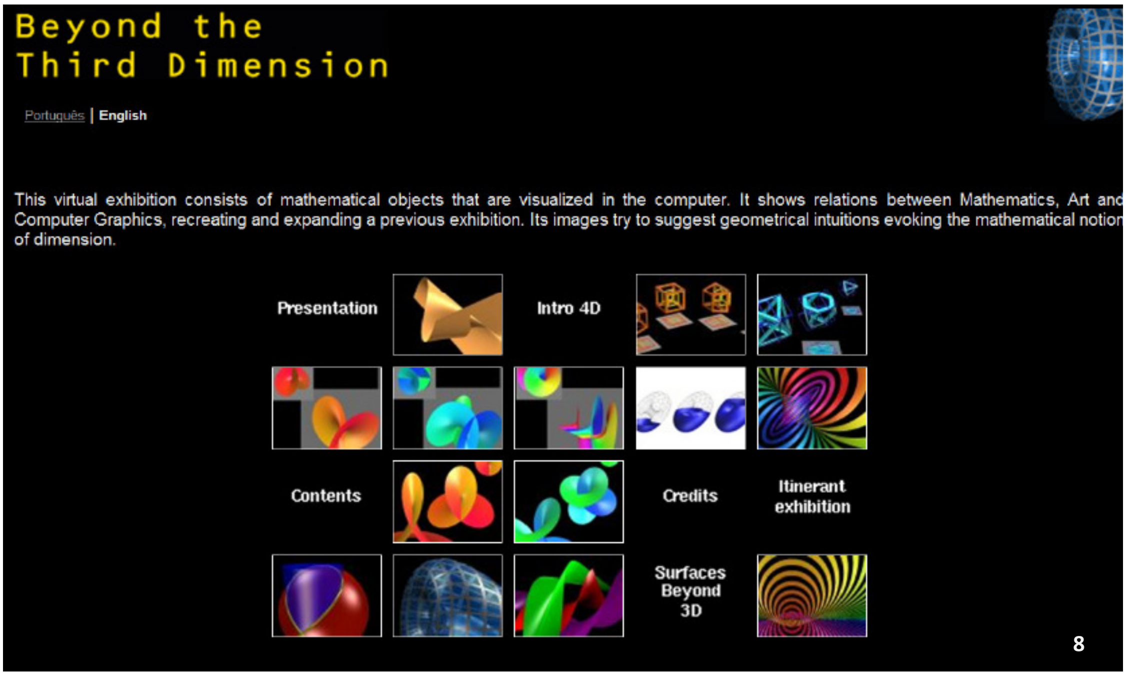
WMY2000 and the Project *Matemática em Acção*


7

During the World Mathematical Year 2000 many research institutions accepted the challenge of dedicating some effort to public awareness activities to help improve the image of mathematics. Answering this call, CMAF decided to initiate a programmatic activity on "mathematics as an educational resource" and launched a project for the popularization and communication of the mathematical sciences called *Matemática em Acção* <w4>. This project (Fig. 7) started essentially with two complementary goals: to enhance the interaction with schools and their teachers through the development of scientific material about mathematics through an innovative approach and the promotion of public awareness activities related for mathematics.

Concerning the first goal, in the year 2000 CMAF produced the Portuguese version of the video series from the California Institute of Technology *Project MATHEMATICS*, by Tom Apostol. This remarkable collection of eight didactical videos, covering topics such as the Pythagorean theorem, similarity, polynomials and trigonometry, introduces mathematical ideas by animated images, capitalizing the viewer's geometric intuition and conveying school mathematics in a rich cultural context [17]. Because these topics are integrated with the pre-university curricula, the Portuguese Ministry of Education supported all costs concerning the production and the free distribution of the Portuguese versions of these videotapes in public schools. Sessions with teachers to improve the use of the videos in the classroom were organized. These sessions showed the importance of providing scientific support to teachers concerning the deeper approaches of the topics included in the videos and to stimulate their use in a non passive way. In the following year the project *Matemática em Acção*, produced and distributed the Portuguese version of Apostol's ninth videotape, on the *Early History of Mathematics*, describing in about 30 minutes some important developments, from Babylonian calendars on clay tablets, to the invention of calculus in the seventeenth century.

Concerning the second objective and to illustrate the role of mathematics in several fields the project started to develop itinerant exhibitions, available to schools and cultural entities upon request. Starting from a virtual exhibition that was the result of an international cooperation between two mathematicians in the United States (T Banchoff and D Cervone) and a local team of mathematicians and computer professionals [18], and developed only through the internet, *Para Além da Terceira Dimensão - Beyond the Third Dimension* (Fig. 8) was recreated as a physical exhibition with the collaboration of the town of Óbidos, starting on 14th October 2000. Banchoff and Cervone came to Portugal for 22nd November for the opening of the exhibition in Funchal, at the University of Madeira. Becoming a traveling exhibition it visited more than twelve institutions in Portugal and crossed the Atlantic to the *Instituto de Matemática Pura e Aplicada,* in Rio de Janeiro, Brazil, in July 2001. It is still alive in cyberspace <w5>. This virtual exhibition, named after a classic book [19] by Thomas F Banchoff, describes twelve surfaces and includes small movies representing aspects of geometrical objects in three and four dimensions, like the “Triple Point Twist”, “Necklaces”, “In and Outside the Torus”, “The Klein Bottle”, “Math Horizon” and two animated pieces on the hypercube, “A Rotation of Cubes” and “Iced Cubes” [20]. These images suggest geometrical intuitions evoking the notion of dimension and illustrate the relationship between mathematics, art and computer graphics. They stimulate the imagination and the curiosity of young people and the public in general, showing the potential of new technology for information and communication, through a fascinating experience that relates mathematical concepts to real and abstract objects. They are one element of a virtual museum of mathematics [21], a vision that several mathematicians started to dream of in the last century and is now becoming, step by step, a reality.



Within the framework of the WMY2000, a Portuguese version of the computer-generated film on the mathematical visualization of minimal surfaces *Touching Soap Films* <w6>, by K Polthier and co-authors, was produced. It was distributed on a non-profit basis, for example, to the high schools. CMAF sponsored the European launch of the CD-ROM *Raising Public Awareness of Mathematics*, a project coordinated by M Chaleyat-Maurel, V L Hansen and R Brown, which took place in the town of Óbidos, on 11 November 2000. CMAF also organized the international workshop on *Multimedia Tools for Communicating Mathematics* that took place in Lisbon [22] in that year.

Two new exhibitions were organized in the town of Óbidos by CMAF, both in collaboration with SPM. The first, *As Sombras do Tempo – The Shadows of Time* [23] is a successful exhibition on sundials that began in that town in June 2002. The other exhibition, the *1st Exhibition of Mathematical Games*, opened in June 2003. This initiative and another one that started in the following year, the Portuguese Championship of Mathematical Games, were at the origin of *Associação Ludus* <w7>. This association, based at the Museum of Science also aims to promote mathematics, emphasizing its cultural, historical and recreational aspects, focusing on mathematical games. The fascinating topic of board games was also the subject of two other exhibitions, organized by the Department of Mathematics of the Faculty of Sciences. The first, in collaboration with the City Museum of the Town of Lisbon, in 2004, with rich ethnological and archaeological components was *Pedras que Jogam* (*Stones that Play*, [24]) <w7>. The second, in 2006 also in Lisbon, in collaboration with the Centro Cultural de Belém was *Matemática em Jogo* and had some computer games on display <w9>.

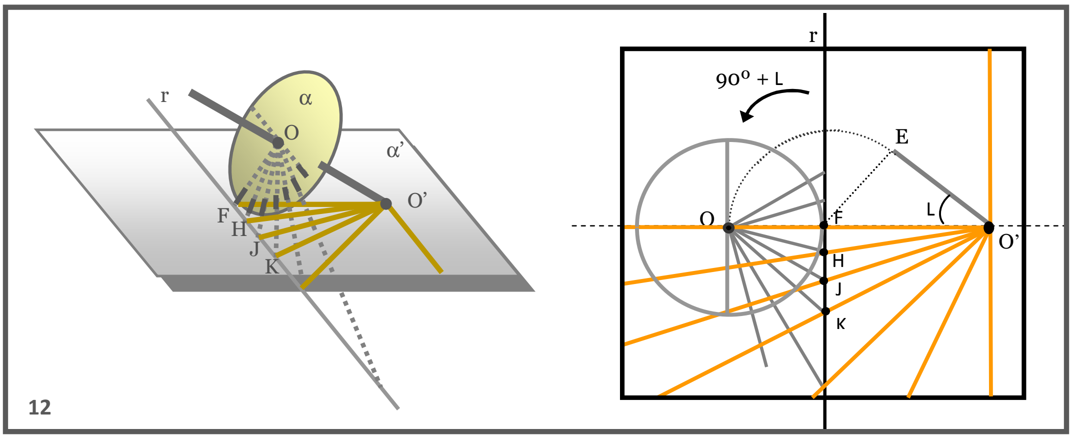
Ruled Surfaces, Sundials and Architecture

Models for designing ruled surfaces and their intersections were used in the course on descriptive geometry, created in 1859 at the Polytechnic School. The Museum of Science lodges a collection of twenty of those models, dating from the 19th century and constructed by Fabre de Lagrange in Paris (Fig. 9). This collection inspired the project *Geometry: to manipulate and to visualize*, which aimed to: i) construct replicas of three of the models; ii) use the replicas for the visualization of the surfaces simulated by the straight lines (cylinder, cone, hyperboloid of one sheet, hyperbolic paraboloid); iii) model the devices using *Mathematica* and to carry out a mathematical study of the surfaces produced by the instruments[5].

Research into ruled surfaces in contemporary architecture was developed and presented in the exhibition *The Collections of the Museum* on show at the Museum of Science from March to September 2009. As emblematic examples, we have the Cathedral of Brasília by Óscar Niemeyer (Fig. 10), which uses a hyperboloid of one sheet, and the Oceanarium in Valencia by Felix Candela (Fig. 11), which uses hyperbolic paraboloids. These surfaces are also used for roofs as by St Mary's Cathedral in San Francisco (Pietro Belluschi and Pier Luigi Nervi) and the roof of the reconversion of the Market of St Caterina in Barcelona (José María Velaso). The Portuguese architect Álvaro Siza also used surfaces having a curve as directrix for the Tent of Portugal pavilion at the EXPO 98, a building that marks the renewed east-end of Lisbon.

Sundials are a good way for students to relate concepts of astronomy and of mathematics together with the history of with these instruments. They also promote an interest in architectural and aesthetics aspects, and so they are important for the rounded development of the individual.

Starting with the most basic sundial, the equatorial one, we can directly draw the hour lines of a horizontal (or vertical) dials by projecting the equally spaced hour lines of an equatorial dial (equally spaced by 15º) onto a horizontal (or vertical) plane (Fig. 12).



12

From a mathematical point of view, we can use basic plane trigonometry to calculate the angles between the hour lines of a horizontal dial constructed for any latitude *L*. Using the rectangular triangles FOO', FHO and FHO' we have:

$$\sin L = \overline{FO}/\overline{FO'},\ \tan 15^\circ = \overline{FH}/\overline{FO} \quad \text{and} \quad \tan \angle FO'H = \overline{FH}/\overline{FO'} = \overline{FO}\ \tan 15^\circ/\overline{FO'},$$

that is,

$$\tan \angle FO'H = (\sin L)\ (\tan 15^\circ).$$

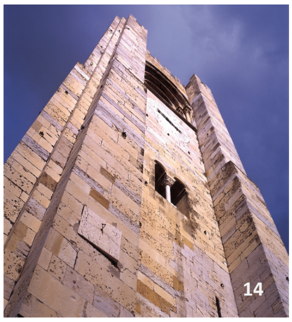
Similarly, as $\angle FOJ = 30^\circ$, $\angle FOK = 45^\circ$, we have:

$$\tan \angle FO'J = (\sin L)\ (\tan 30^\circ),\ \ \tan \angle FO'K = (\sin L)\ (\tan 45^\circ) \text{ and so forth.}$$

For vertical sundials the formulas are analogous, using *cos L* , if the plane of the hour lines faces south, but can be more complicated if this plane is not perpendicular to the north-south direction.


13

These simple properties led to the development of the already mentioned itinerant exhibition *As Sombras do Tempo - The Shadows of Time* (Fig. 13), based on the work developed with prospective teachers concerning the connections between astronomy and mathematics[6]. This exhibition was first shown in Óbidos during the interdisciplinary conference *Nexus 2002: Relationships between Architecture and Mathematics*. This conference, the fourth of a series that started in Italy, was co-organized by the project *Matemática em Acção* and contributed to the identification of the mathematical principles that are used, on one hand, as a basis for architectural design or as tools for analysing existing monuments and, on the other hand, to see architecture as a concrete expression of mathematical ideas or, in a sense, as "visual mathematics" through history and among different cultures [25].


14

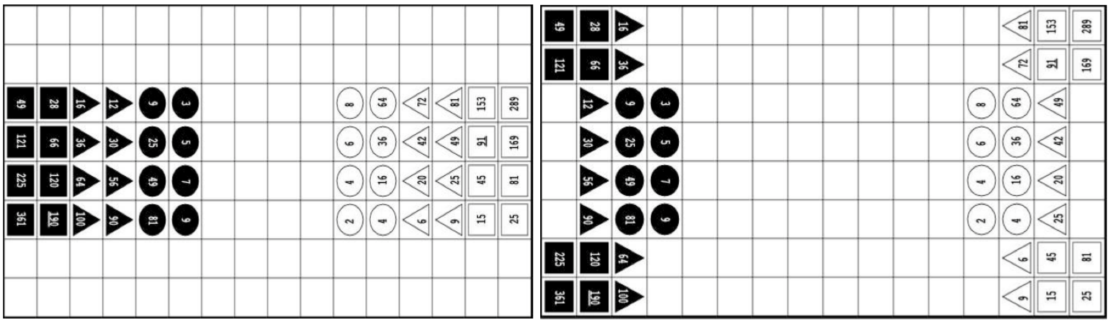
The interest that it raised and the multiple requests led to a new exhibition, *Sundials and Mathematics,* which included more scientific details and images of Sundials in Lisbon (Fig. 14). All those activities were coordinated within *The Shadows of Time* project <w11>, in the framework of *Matemática em Acção,* which develops interactions between several levels of teaching using sundials. While contributing to the awareness of sundials in Portugal, this project intends to help preserve the national heritage by encouraging the restoration of these monuments, some of them totally abandoned in recent decades [26]. It also gives scientific support to the design of dials, as for instance, the collaboration between S Nápoles, N Crato and F Correia de Oliveira concerning the hour lines of the biggest sundial in Portugal (Fig. 15), in the front of the Tempus International building (authorship Tetractys-Architects). In this example of contemporary architecture, there is no shadow projected by a gnomon to indicate the hours. It is a line of light that indicates the hours. The declination of the wall containing the hour lines imposed the type of the dial, an oriental one. In this case, the angle z between an hour line and the noon line can be obtained using the corresponding angle H in the equatorial sundial with the same gnomon and the formula $\tan z = \cos L(\cos d \cot H + \sin d \sin L)$ where L is the latitude and d (=7,78º) is the declination of the plane of the dial.

Going further with Mathematical Games

Games and puzzles, whose practice resembles the mathematical way of thinking, in one way or another, have been known throughout history, and were surveyed recently [27]. The interaction of games and puzzles with recreational mathematics has a long tradition and shows kinship with mathematics and logic. This interaction has stimulated not only the invention of computer mathematical games such as, for example, OZ (tilings of polygons by lozenges [28]), but also mathematical challenges, like the complete solution of the problem for characterizing a class of patterns for a solitaire game formulated by Maxim Kontsevich [29]. This interaction was shown, in particular, at two colloquia on recreational mathematics (see <w12> and <w13>), that were recently organized by *Associação Ludus,* with the support of the Museum of Science.

How can we say that one board game is more mathematical than another? The answer to this question is not easy to find. However, it is not difficult to agree that chess and go are more mathematical than Monopoly. The connections with mathematics are diverse and can be obvious or lie hidden in structure of the rules. To illustrate this fact, consider two games that deserve special attention: Rithmomachia and Hex. The first is the oldest didactical game known [30]. Created in the 11th century to teach the arithmetic of the quadrivium, it was known by every educated man in Europe until the 16th century. It was used in the classroom, first in the monasteries and later in the universities, and only when the mathematical tradition changed did this game, sharing the fate of the quadrivium, vanish. Hex was independently invented twice in the 1940s and is the paradigm of a new kind of board game: a connection game [31]. The fact that one of its inventors was the Nobel laureate John Nash, besides having been popularized by Gardner, adds to its relevance. It has been noticed that there is a profound connection between Hex and higher mathematics.

Rithmomachia unfolds on a chequered 8 x 16 board, where two adversaries - White (even) and Black (odd) - control their armies of 24 numbered pieces (Fig. 16).



16

The starting position is determined by several mathematical relations and proportions. We explain how to implement the first of the two positions shown. The second, the starting position in Rithmomachia, is easily derived from the first. We treat the disposition of the white pieces; the black ones follow a similar pattern. Each piece has a number, and below we show how the pieces are laid out in the first position, with modern notation to the right.

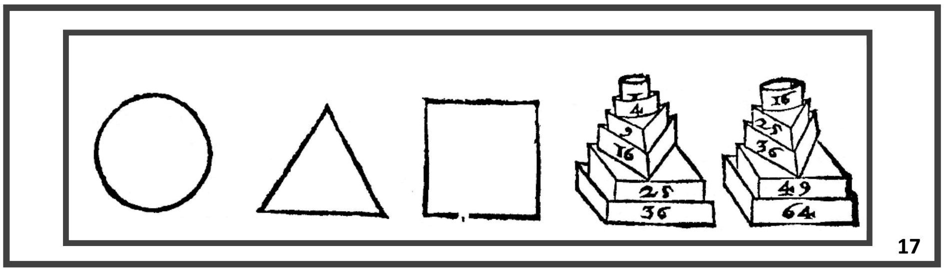
From the even numbers 2, 4, 6, 8 placed on circular pieces in the central cells of the sixth row, we deduce the remaining numbers. The second row, with circular pieces, has the squares of the numbers in the first:

2	4	6	8	n
4	16	36	64	n^2

The following row, with triangular pieces, gets its numbers by addition of the previous two. The next row also has triangular pieces. Each piece has a number that is the square of one more than the number on the first circular piece. The pieces of the first row of square shapes get their numbers from addition of the previous two. All these operations are illustrated below:

2	4	6	8	n
4	16	36	64	n^2
6	20	42	72	$n(n+1)$
9	25	49	81	$(n+1)^2$
15	45	91	153	$(n+1)(2n+1)$
25	81	69	289	$(2n+1)^2$

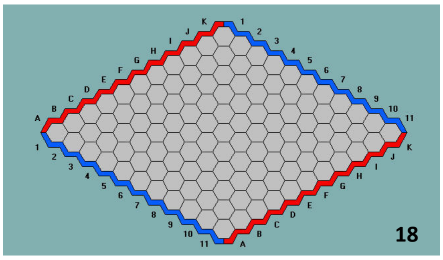
There are two pieces (91 and 190), one of each colour, that are replaced by special solid objects - the pyramids (Fig. 17). These are sums of square numbers, each corresponding to one of the described shapes. In the version of the rules that we are considering, the white pyramid has six stories, numbered 1, 4, 9, 16, 25 and 36. The black one has five: 16, 25, 36, 49 and 64. Note that 1+4+9+16+25+36 = 91 and 16+25+36+49+64 = 190.



17

Movement is easy: discs move orthogonally one square, triangles two cells diagonally, while squares move three steps, orthogonally. Pyramids move as the chess queen, but are limited to four squares. Captures can happen in several ways. For instance, captures relating to addition and subtraction: if two pieces belonging to the same player can move to a cell occupied by an adversary piece and if the number of this piece is equal to the sum or difference of the numbers of the pieces of the first player, the piece is captured. There is no replacement.

A player wins when he places, in the adversary's half of the board, three pieces in any of the types of mathematical progression. For instance 2-15-28 is an arithmetic progression, 9-15-25 is a geometric one, and 9-15-45 is a harmonic progression. Victory can also be achieved by placing four pieces showing a combination of two progressions, as 2-3-4-8, a combination of arithmetic (2-3-4) and geometric (2-4-8) progressions, or even three, as in 4-6-9-12.



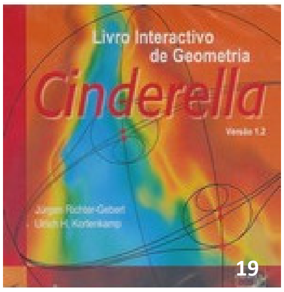
18

Hex (Fig. 18) was invented twice in a short period of time first by Piet Hein, a Danish scientist and poet, in 1942, and by the mathematician John Nash in 1947. The rules of Hex are surprisingly simple: players alternate placing counters of two different colours, trying to connect parallel margins of the board. The game cannot be drawn, and this can be seen in several ways, of varying formal presentation. If we assume that the board is filled with red and blue pieces, and think of blue as water and red as one foot-high walls and bricks, then either water flows between the blue margins (blue wins) or there is a red wall preventing blue communication (red wins). Proofs using graph theory and by induction can be found in the literature. This result is known as the Hex theorem.

Nash gave a beautiful non-constructive proof that the first player must have a winning strategy in Hex. His argument - the now famous *strategy stealing argument* - goes as follows: as there are no ties, either the first or second player must have a winning strategy. Suppose it is the second. Then the first player makes his first move at random and assumes the role of the second player. The reason this trick works is that, as Hex is a connection game, an extra move, no matter how bad it might be, cannot hurt its player. Therefore, with this plan, the first player wins the game, which gives us a contradiction. We must conclude that the first player has a winning strategy. If the size of the board is large enough, nobody knows how to play perfectly, so the practice of Hex is not ruined by Nash's result.

The connection between Hex and mathematics was emphasized when David Gale, a mathematician from Berkeley, showed that the Hex theorem is logically equivalent to Brouwer's fixed-point theorem [32], a deep result in topology. Knowing this equivalence does not increase your playing strength, but shows how abstract games and mathematics are linked in a mysterious, fascinating way.

Mathematics as an Educational Resource



At the 2000 conference on *Multimedia Tools for Communicating Mathematics* organized by CMAF in Lisbon, the authors of the interactive geometry software *Cinderella* presented the theoretical background that is required to build a consistent and continuous foundation for dynamic elementary geometry [33]. This helped the project *Matemática em Acção* to develop a further collaboration with the Ministry of Education, who sponsored the Portuguese translation and edition of the dynamic geometry software *Cinderella*, as well as its manual (Fig. 19). Every secondary school in Portugal received in 2001 a free copy of the interactive book [34]. A virtual forum was created on the web, containing hundreds of applets and experiences, which are shared through the *Cinderella* Portuguese website <w14>. That conference also allowed the project *Matemática em Acção* to develop a collaboration with the Computer Science Department of the Faculty of Sciences of the University of Lisbon. This collaboration meant that the concept of hypervideos could be added to the multimedia tools for communicating and learning mathematics [35]. A first demo was developed based on the video *The Story of Pi*, by Apostol [36], and presented in a mini-symposium at ICIAM 2003, in Sydney, and at the National Meeting of Scientific Visualization held in Espinho, Portugal, in 2005. Finally, the complete interactive CD-ROM [37] was published in Portuguese in 2007. Although based on the components of the Portuguese translation of the original Apostol film, the final hypervideo has an original structure and interactive navigation, which goes far beyond the limited capability of video supporting learning and has attached a new virtual booklet with the history of the number *Pi*. Hypermedia gives the user greater control and autonomy, so he can explore the links to the information which is conveyed by the video and complemented by other material, augmenting its capabilities as a cognitive artifact. Hypermedia allows the exploration of topics at different levels of knowledge with several levels of complexity, from elementary topics like the calculation of areas and volumes to more sophisticated concepts like random phenomena, approximate calculation and Fourier series. This makes the hypervideo *A História do Pi* (Fig. 20) a useful tool for teacher training.

The use and the development of the concept of hypermedia was successfully proposed to the University of Madeira for the training of mathematics teachers and has led to three master's theses, including one that developed a plan for integrating the three Apostol videos on trigonometry into a single hypervideo adapted to the Portuguese secondary school curricula [38].

The study of the applications of Mathematics has been revealed as one of the most efficient ways to motivate the apprenticeship of this science. A comprehensive foundation, as well as the establishment of connections, is essential in education. Bearing in mind the importance of providing materials and resources for teachers, several have been produced [39], [40], and an interactive book is currently under development[7]. The interactive online book *Studying Mathematics with Spreadsheet* <w15>, includes links to computational applications and activities, designed to help teachers in almost all subjects. The design seeks to identify concepts and problems that students struggle with, and complement them with appropriate technology-based tools. The resulting tools vary in terms of their interactivity, but they are closely related to conceptual ideas presented in the textbooks and always raise questions for the students and the teacher to explore.

Final remarks

Although the popularization and teaching of mathematics are related they are different components of the communication of mathematics, and it is clear that there are important differences between them. For example in their aims: the purpose of popularization is to raise awareness, not just to educate, and the criterion of success is not only an increase of knowledge, but also a change of attitude [1]. However, new tools and new media provide new possibilities and raise new challenges in the communication of mathematics to different audiences, from pupils to teachers, from other scientists to the general public, requiring innovative ideas and interdisciplinary collaboration.

References

[1] A.G. Howson and J.-P. Kahane (Editors), *The Popularization of Mathematics,* ICMI Study #5, Cambridge Univ. Press, Cambridge, 1990.

[2] F. Bragança Gil, *Museu de Ciência da Universidade de Lisboa: das origens ao pleno reconhecimento*, MCUL, Lisboa, 2003.

[3] V. Rivotti Silva, *L'Observatoire Astronomique*, in [7]: Museums of Science and Technology, *Fundação Oriente, Lisboa,* (1998), 125-130.

[4] M.C. Lourenço and A.M.Eiró, *O Museu de Ciência*, in pp. 35-54 of *Património da Universidade de Lisboa – Ciência e Arte* (Coord. M.C. Lourenço and M.J.Neto), Ed. Tinta da China, Lisboa, 2011.

[5] *José Anastácio da Cunha* (1744-1787), *Matemático e Poeta,* Catalogue of the exhibition. Biblioteca Nacional, Lisboa, 1987.

[6] *Movimento Matemático 1937 – 1947*, Catalogue of the exhibition commissioned by Ilda Perez. Câmara Municipal de Lisboa, 1997.

[7] M. Alzira Ferreira and J.F. Rodrigues (Editors), *Museums of Science and Technology,* Fundação Oriente, Lisboa, 1998.

[8] A. Beutelspacher, *Towards a Mathematics Museum*, in [7]: Museums of Science and Technology, *Fundação Oriente*, Lisboa, (1998), 155-163.

[9] M. Arala Chaves, *Expositions Interactives de Mathématiques,* in [7]: Museums of Science and Technology, *Fundação Oriente*, Lisboa, (1998), 165-188.

[10] *A matemática e a música,* Colóquio/Ciências, Revista de Cultura Científica, nº24, November 1999, *Fundação Calouste Gulbenkian*, Lisboa.

[11] G. Assayag, H.D. Feichtinger and J.F. Rodrigues (Editors), *Mathematics and Music, A Diderot Mathematical Forum,* Springer-Verlag, Berlin Heidelberg 2002.

[12] J. Rezende, L. Monteiro, E. Amaral, *António Aniceto Monteiro, uma fotobiografia a várias vozes,* Sociedade Portuguesa de Matemática, Lisboa, 2007

[13] *Proceedings of the Board Game Studies Colloquium* XI, J.N. Silva Ed., Associação Ludus, Lisbon 2009.

[14] A.Carvalho, C Santos, J. Neto, J.N. Silva, *Jogos Matemáticos, A Portuguese Project*, in [13], pp. 19- 25

[15] L. Saraiva, *History of Astronomy in Portugal: Institutions, Theories, Practices*, in p. 32-34, CIM Bulletin nº27, Jan 2010.

[16] *Medir os Céus para dominar a Terra,* catalogue of the exhibition, Ed. Museu de Ciência da Universidade de Lisboa, 2009

[17] T. M. Apostol, Computer Animated Mathematics Videotapes, in [22], pp. 1-27.

[18] T.F.Banchoff and D.P.Cervone, *A Virtual Reconstruction of a Virtual Exhibit*, in [23], pp. 29-38.

[19] T.F.Banchoff, *Beyond the Third Dimension: Geometry, Computer Graphics, and Higher Dimensions,* Scientific American Library, W.H.Freeman and Co., New York 1990.

[20] *Para Além da Terceira Dimensão - Beyond the Third Dimension* Catalogue of the exhibition by CMAF/Universidade de Lisboa, 2000.

[21] J.F.Rodrigues, *Um Museu Virtual de matemática – um desafio real para o ano 2000!* Boletim da SPM, nº34, Maio 1996, pp.77-79.

[22] J.Borwein, M.H.Morales, K.Polthier, J.F.Rodrigues (Eds.) *Multimedia Tools for Communicating Mathematics,* Springer-Verlag, Berlin-Heidelberg, 2002.

[23] *"As Sombras do Tempo – The Shadows of Time"*, Catalogue of the exhibition on Sundials co-ordinated by Suzana Nápoles et al. CMAF/Universidade de Lisboa, 2002.

[24] *"Pedras que Jogam",* Catalogue of the exhibition commissioned by Adelaide Carreira et al., DM/Faculdade de Ciências/Universidade de Lisboa, 2004.

[25] J.F.Rodrigues and K. Williams (Eds.), *NEXUS IV Architecture and Mathematics*, KWBooks, Fucecchio (Florence), 2002.

[26] N.Crato, S. Nápoles and F.C. Oliveira, *Relógios de Sol,* CTT/Correios de Portugal, Lisboa, 2007.

[27] Silva, J.N., *"On Mathematical Games"*, BSHM Bulletin, Journal of the British Society for the History of Mathematics, 26:2 (2011), 80-104.

[28] A.P.Cláudio, et al. *"Implementing L'OZ",* in pp. 51-58 of Proceedings of the Recreational Mathematics Colloquium I, J.N. Silva (Ed.), Associação Ludus, Lisboa 2010
[29] P. Duarte, *"Solution to a Solitaire game"*, in pp. 161-171 of Proceedings of the Recreational Mathematics Colloquium I, J.N. Silva (Ed.), Associação Ludus, Lisboa 2010.
[30] A. E. Moyer, *The Philosophers' Game: Rithmomachia in Medieval and Renaissance Europe,* The University of Michigan Press 2001
[31] C. Browne, *Hex Strategy: Making the Right Connections,* AK Peters 2000.
[32] Gale, D., *"The Game of Hex and the Brouwer Fixed-Point Theorem"*, American Mathematical Monthly, 86 nº10 (1979), 818-827.
[33] U.H.Kortenkamp and J. Richter-Gebert, *A Dynamic Setup for Elementary Geometry,* in [16], pp.203-219.
[34] U.H.Kortenkamp and J. Richter-Gebert, *Cinderella: Livro Interactivo de Geometria* (Versão 1.2; Trad. J.N. Silva), CMAF/Universidade de Lisboa, 2001.
[35] T. Chambel and N. Guimarães, *Communicating and Learning Mathematic with Hypervideo,* in [16], pp.79-91
[36] T.M. Apostol, *The Story of Pi,* Project MATHEMATICS, Caltech, 1989.
[37] T.M.Apostol, T.Chambel, S.Nápoles, J.F.Rodrigues, L.Santos, *A História do Pi em hipervideo,* CD-ROM Interactivo, Textos Editores, Lisboa, 2007
[38] Sandra Sousa, *Contribuição para um Hipervídeo na Trigonometria,* Master Thesis in Educational Mathematics (Directed by J.F.Rodrigues), Universidade da Madeira, Funchal, 2002.
[39] M.C.Oliveira and S. Nápoles, *Using a Spreadsheet to Study the Oscillatory Movement of a Mass-Spring System*, Spreadsheets in Education (eJSiE), Bond University, Australia, 2010
[40] M.C.Oliveira and S. Nápoles, Sundial,s *Mathematics and Astronomy with a Spreadsheet,* ICERI 2010, International Conference of Education, Research and Innovation, Madrid, 15-17 Nov 2010.
[41] J. Rezende, *Puzzles with Polyhedra and Numbers*, in [13] pp. 105-133.

Figures and Captions

Fig. 1 The 19th century building of the Polytechnic School, later the Faculty of Sciences, hosting today two Museums of the University of Lisbon: Science and Natural History
Fig. 2 The Astronomical Observatory of the Polytechnic School (1898), © Mark Heller
Fig. 3 Spherical triangle calculator. Observatório Astronómico da Faculdade de Ciências da Universidade de Lisboa, MCUL547
Fig. 4 Rithmomachia in *Mathematical games throughout the ages*: an exhibition at the MCUL, Lisbon
Fig. 5 Circle of proportions, Elias Allen, London, c. 1630, MCUL501
Fig. 6 *Measuring time, measuring the world, measuring the sea*: part of the exhibition *Measuring the sky to rule the territory*, at the MCUL, in Lisbon
Fig. 7 Logo of the project *Matemática em Acção*, 2000.
Fig. 8 *Para Além da Terceira Dimensão - Beyond the Third Dimension*
Fig. 9 A model from the 19th century for designing ruled surfaces and their intersections
Fig. 10 Cathedral of Brasília
Fig. 11 Oceanarium in Valencia
Fig. 12 How to draw the hour lines on a horizontal dial
Fig. 13 *As Sombras do Tempo* (The Shadows of Time)
Fig. 14 Sundial at Sé de Lisboa (Lisbon Cathedral)
Fig. 15 The biggest sundial in Portugal
Fig. 16 Schematic representation of Rithmomachia's board
Fig. 17 Rithmomachia's pieces in Barozzi's manual
Fig. 18 Empty Hex board
Fig. 19 Cinderella: an interactive book of geometry (in Portuguese)
Fig. 20 *The history of Pi* in hypervideo (in Portuguese)

Web Sites

1 http://cmaf.ptmat.fc.ul.pt/
2 http://www.atractor.pt/
3 http://www.mathex.org/
4 http://wwmat.ptmat.fc.ul.pt/em_accao/
5 http://alem3d.obidos.org
6 http://page.mi.fu-berlin.de/polthier/video/Touching/index.html
7 http://ludicum.org
8 http://mat.fc.ul.pt/divulgacao/pedrasquejogam/ inaugural
9 http://mat.fc.ul.pt/divulgacao/mej/
10 http://wwmat.mat.fc.ul.pt/em_accao/superficies_regradas/
11 http://sombrasdotempo.org/
12 http://ludicum.org/rm09
13 http://ludicum.org/rm11
14 http://cinderella.ptmat.fc.ul.pt/
15 http://matematicainteractiva_dm.fc.ul.pt

Footnotes

[1] The historic building of the Museum also hosted the first editorial office of *Portugaliae Mathematica*, the Portuguese mathematics research journal founded in 1937, the *Centro de Estudos Matemáticos de Lisboa* and also the first editorial office of the *Gazeta de Matemática*, both dating from 1940.

[2] These proceedings also include an article on the Beijing astronomical observatory, which has more than 550 years of history and possesses unique bronze mathematical instruments of the Ming dynasty (17th century), which have recently been restored and constitute an open air museum.

[3] The Portuguese component of the Diderot Mathematical Forum was organized by CMAF and was partially held at the University of Lisbon in co-ordination with the XII National Seminar on the History of Mathematics organized by SPM.

[4] To support these activities, in addition to traditional mathematical puzzles, new materials have been produced as, for instance, an original puzzle with polyhedra and numbers exploring permutation groups invented by Jorge Rezende [41].

[5] This work was carried out in partnership with the S. João de Brito School and was presented in the 4th Forum *Ciência Viva*, held in Lisbon in May 2000 <w10>.

[6] This initiative was developed as a collaboration between the high school *António Arroio* and the Department of Mathematics, Faculty of Science of the University of Lisbon.

[7] This interactive book is in development through a partnership between the project *Matemática em Acção*, the Department of Mathematics, Faculty of Science of the University of Lisbon and a Basic School (EB2,3 Piscinas, Lisbon).